Homological Mirror Symmetry for Singularities

Takahashi, Atsushi

代数幾何学シンポジウム記録
2008年度 pp.98-101

HOMOLOGICAL MIRROR SYMMETRY FOR CUSP SINGULARITIES

ATSUSHI TAKAHASHI

1. Statement and the result

We associate two triangulated categories to a triple $A := (\alpha_1, \alpha_2, \alpha_3)$ of positive integers called a *signature*: the bounded derived category $D^b\mathrm{coh}(X_A)$ of coherent sheaves on a weighted projective line $X_A := \mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}$ and the bounded derived category $D^b\mathrm{Fuk}^{\rightarrow}(f_A)$ of the directed Fukaya category for a "cusp singularity" $f_A := x^{\alpha_1} + y^{\alpha_2} + z^{\alpha_3} + q^{-1}xyz$, $(q \in \mathbb{C}^*)$. Here, we consider f_A as a *tame polynomial* if $\chi_A := 1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 - 1 > 0$ and as a *germ* of a holomorphic function if $\chi_A \leq 0$.

Then, the *Homological Mirror Symmetry (HMS) conjecture* for cusp singularities can be formulated as follows:

Conjecture 1.1 ([T1]). *There should exist an equivalence of triangulated categories*

$$D^b\mathrm{coh}(X_A) \simeq D^b\mathrm{Fuk}^{\rightarrow}(f_A).$$

□

Combining results in [GL] with known results in singularity theory, one can easily see that the HMS conjecture holds at the Grothendieck group level, i.e., there is an isomorphism

$$(K_0(D^b\mathrm{coh}(X_A)), \chi + {}^t\chi) \simeq (H_2(Y_A, \mathbb{Z}), -I),$$

where Y_A denotes the Milnor fiber of f_A.

The HMS conjecture is shown if $\alpha_3 = 1$ (Auroux-Katzarkov-Orlov [AKO], Seidel [Se1], van Straten, Ueda, ...). Also the cases $A = (4,4,2), (6,3,2)$, which correspond to two of three simple elliptic hypersurface singularities, are known ([AKO], [U], [T2], ...).

The following is our main theorem:

Theorem 1.2. *Assume that $\alpha_3 = 2$ or $A = (3,3,3)$. Then the HMS conjecture holds.* □

The keys in our proof are; the reduction of surface singularities to curve singularities (the stable equivalence of Fukaya categories given in [Se2] section 17), the use of A'Campo's divide [A1][A2] in order to describe the Fukaya category, and mutations of

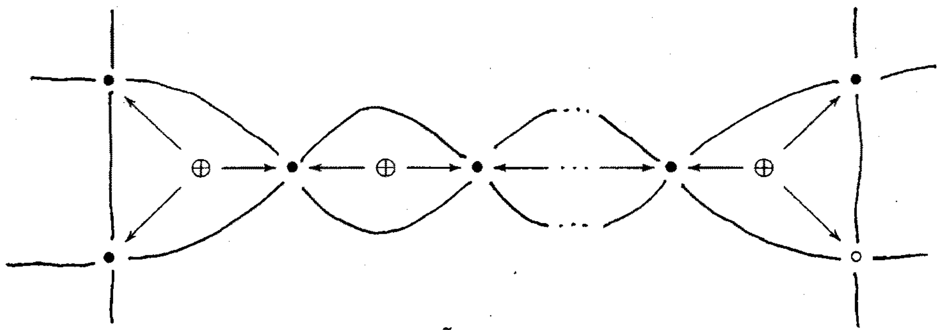
Date: January 27, 2009.

exceptional collections (distinguished basis of vanishing Lagrangian cycles). We shall give devides for cusp singularities with $\alpha_3 = 2$ and also quivers with relations associated to them.

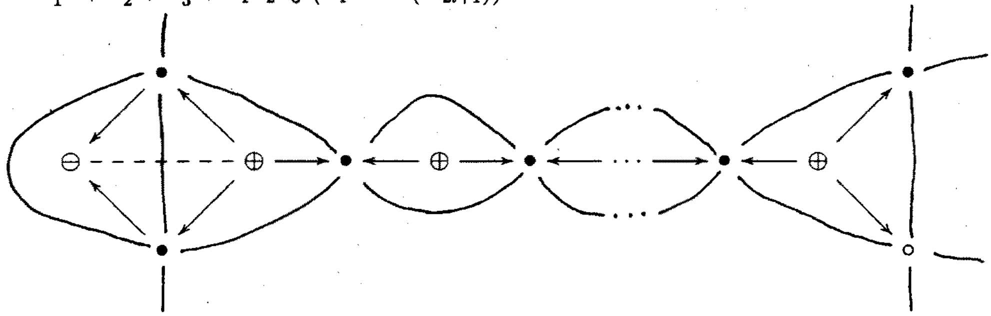
2. Devides and quivers with relations

2.1. $\chi_A > 0$. After applying suitable mutations, we shall obtain the *extended Dynkin quiver* of type $A = (\alpha_1, \alpha_2, \alpha_3)$ ($\circ$ denotes the vertex to remove in order to get the Dynkin quiver of the same type). It is known by [GL] that $D^b\mathrm{coh}(X_A)$ is equivalent to the derived category of extended Dynkin quiver of type A.

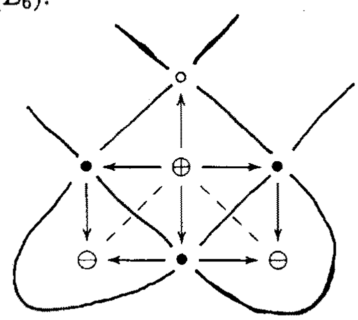
$x_1^{\alpha_1} + x_2^2 + x_3^2 + x_1x_2x_3$ (α_1:even ($\tilde{D}_{2l}$)):



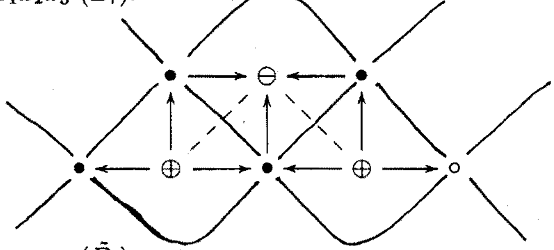
$x_1^{\alpha_1} + x_2^2 + x_3^2 + x_1x_2x_3$ (α_1:odd ($\tilde{D}_{2l+1}$)):



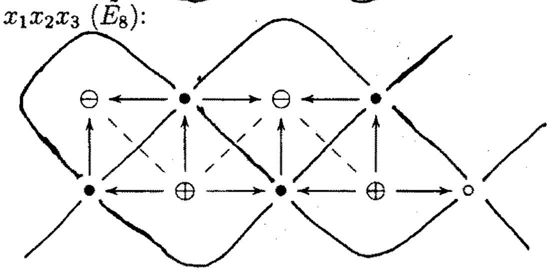
$x_1^3 + x_2^3 + x_3^2 + x_1x_2x_3$ ($\tilde{E}_6$):



$x_1^4 + x_2^3 + x_3^2 - x_1x_2x_3$ ($\tilde{E}_7$):

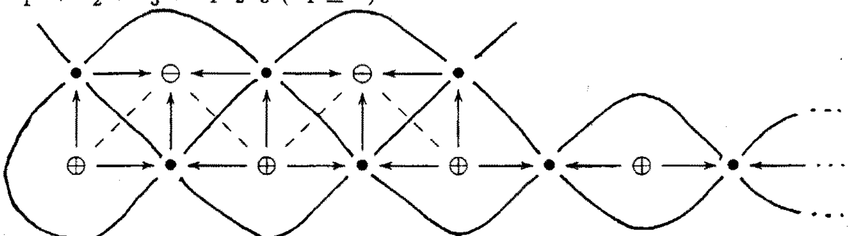


$x_1^5 + x_2^3 + x_3^2 - x_1x_2x_3$ ($\tilde{E}_8$):

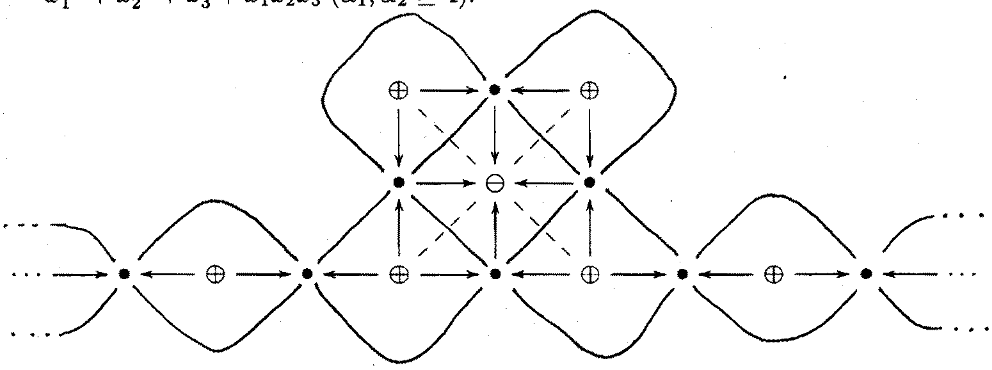


2.2. $\chi_A \leq 0$. Note that the number of vertices (= *Milnor number* of the singularity) is given by $\alpha_1 + \alpha_2 + \alpha_3 - 1$.

$x_1^{\alpha_1} + x_2^3 + x_3^2 + x_1x_2x_3$ ($\alpha_1 \geq 6$):



$x_1^{\alpha_1} + x_2^{\alpha_2} + x_3^2 + x_1x_2x_3$ ($\alpha_1, \alpha_2 \geq 4$):



References

[A1] N. A'Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes I*, Math. Ann. **213** (1975), 1-32.

[A2] N. A'Campo, *Real deformations and complex topology of plane curve singularities*, Annales de la faculte des sciences de Toulouse Ser. 6, 8 no. 1 (1999), 5-23.

[AKO] D. Auroux, L. Katzarkov and D. Orlov, *Mirror symmetry of weighted projective planes and their noncommutative deformations*, Annals of Mathematics, **167** (2008), 867 - 943.

[GL] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), 9–34, Lecture Notes in Math., 1273, Springer, Berlin, 1987.

[KST1] H. Kajiura, K. Saito and A. Takahashi, *Matrix Factorizations and Representations of Quivers II: type ADE case*, math.AG/0511155, Adv. in Math. **211**, 327–362 (2007).

[KST2] H. Kajiura, K. Saito and A. Takahashi, *Triangulated Categories of matrix Factorizations for regular systems of weights with* $\varepsilon = -1$, arXiv:0708.0210.

[LP] H. Lenzing and J. A. de la Pena, *Extended canonical algebras and Fuchsian singularities*, arXiv:math/0611532.

[O] D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, arXiv:math/0503632.

[Se1] P. Seidel, *More about vanishing cycles and mutation, Symplectic Geometry and Mirror Symmetry*, Proceedings of the 4th KIAS Annual International Conference, World Scientific, 2001, 429-465.

[Se2] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*, to appear in the ETH Lecture Notes series of the European Math. Soc.

[T1] A. Takahashi, *Weighted Projective Lines Associated to Regular Systems of Weights of Dual Type*, arXiv:0711.3906.

[T2] A. Takahashi, *Homological Mirror Symmetry for Isolated Hypersurface Singularities in Dimension One*, in preparation.

[T3] A. Takahashi, *Weighted Projective Lines Associated to Maximally Graded Isolated Hypersurface Singularities*, arXiv::0711.3907.

[U] K. Ueda, *Homological Mirror Symmetry and Simple Elliptic Singularities*, arXiv:math/0604361.

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka Osaka, 560-0043, Japan

E-mail address: takahashi@math.sci.osaka-u.ac.jp